Modeling

CSC398 Autonomous Robots

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Outline

1. Modeling and state estimation
2. Examples
3. State estimation
4. Probabilities
5. Bayes filter
6. Particle filter

Modeling

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- The model can never be exact.
- Robots can only estimate states using probabilistic methods for example.

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- Modeling in our soccer agent
  - Ball tracking, opponent localization (and teammates), self-localization, orientation estimation (upright vector).

Examples

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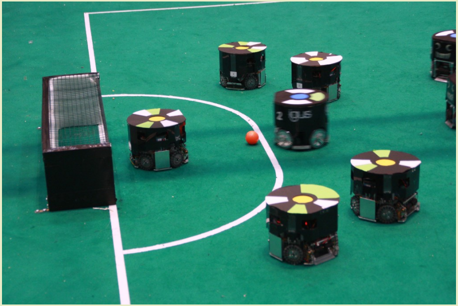
- How can a state estimation be robust despite all the errors?

Example 1

RoboCup Small-Size League:



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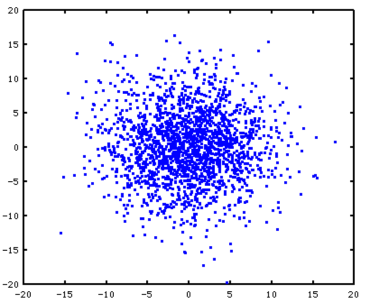

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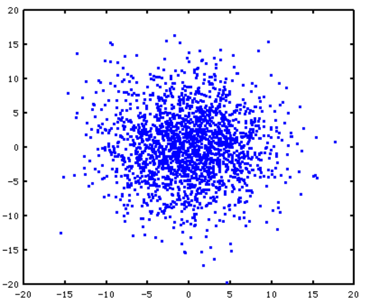

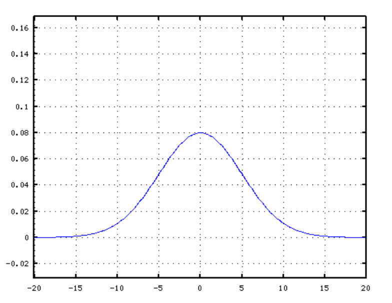

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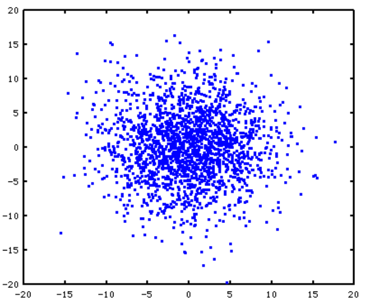

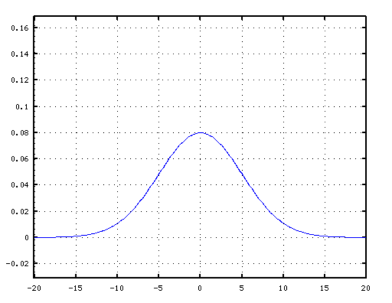

- Problem with two robots: wrong perceptions on other robot.

Example 2

Obstacle avoidance using a laser range finder:

- There can be several different errors in the measurements.

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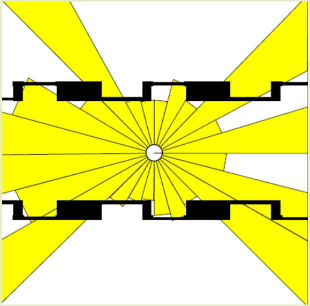

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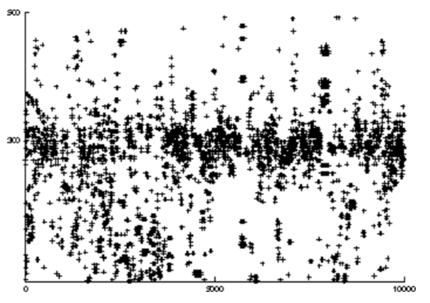

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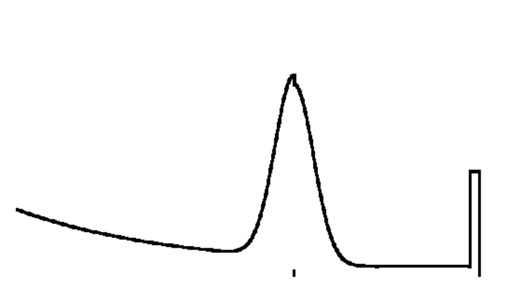

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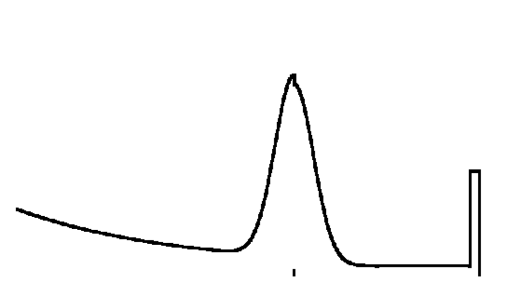

- Knowledge about the behavior of a sensor (the *sensor model*) is very important for a robust state estimation.

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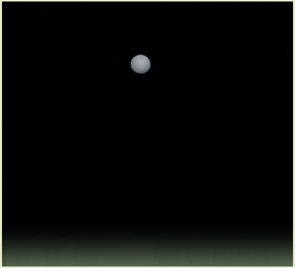

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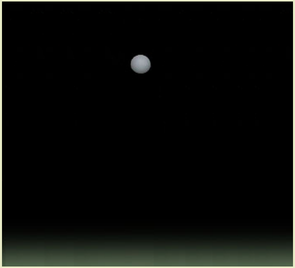

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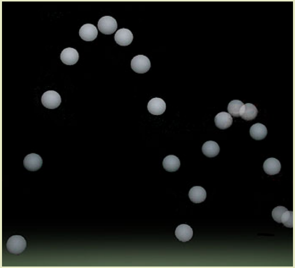

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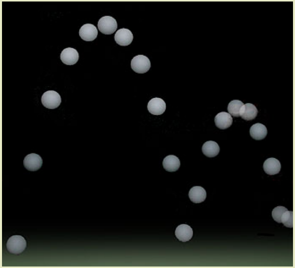

- Knowledge about the behavior of the ball and physics is useful (*state transition model*).

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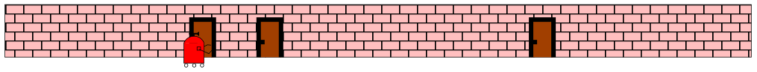

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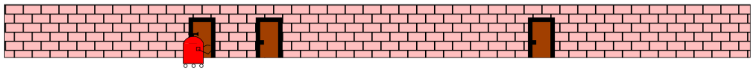

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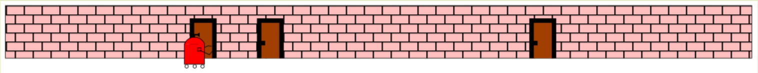

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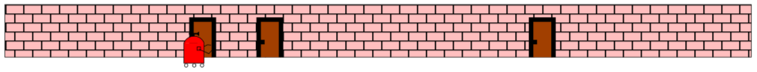

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- Measurements u_t needed (*odometry motion model*).

General state estimation

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- The state estimation gets a sequence of measurements, so the estimation of X_t is based on all measurements $z_0, \ldots, z_t$ and $u_0, \ldots, u_t$.

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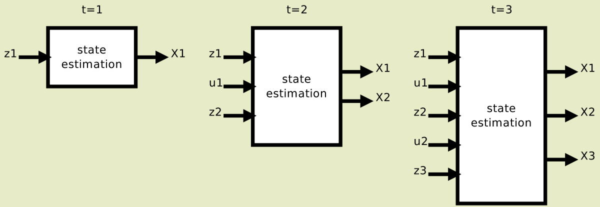

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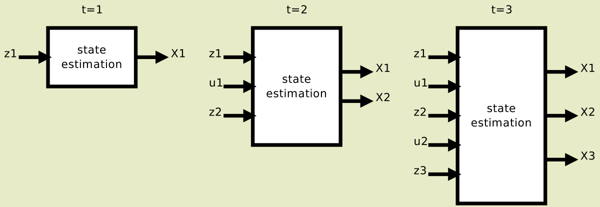


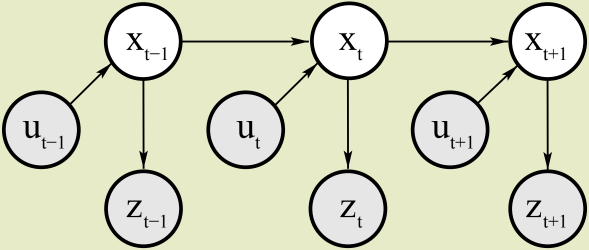
- Problems: more measurements with every time time step $\rightarrow$ increasing amount of computation.

Markov assumptions

- Markov assumption 1:
  The measurement z_t depends only on the state X_t and a random error.
- Markov assumption 2:
  The state transition measurement u_t only depends on the states X_t and X_{t+1} and a random error.

Markov process

Bayesian network with the measurements u_t and z_t:



- The states x_t are hidden.

Recursive state estimation / filter

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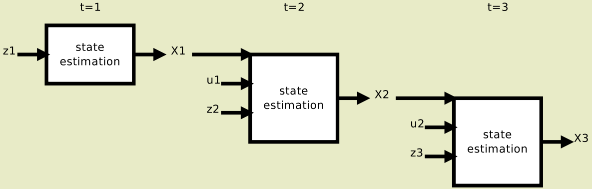

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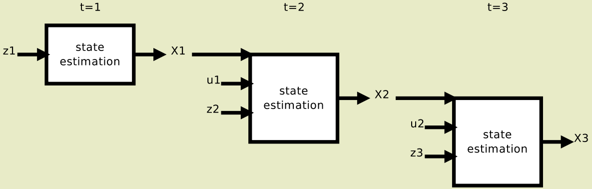

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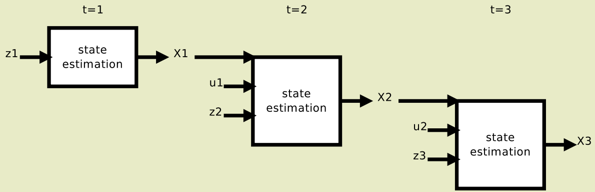

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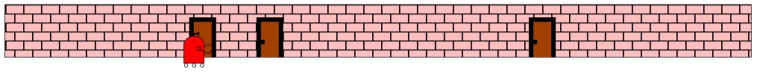
- Needed for X_t is only X_{t-1}, z_t and u_t.
- Belief X_t is updated using only the new measurements $\rightarrow$ constant time for each step.

State estimation

- Sensor model and state transition model needed.
- Update belief X_t using
  - z_t and sensor model.
  - u_t and motion model and knowledge about dynamics in the environment.

Example state estimation



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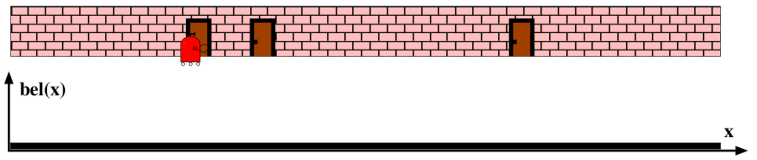

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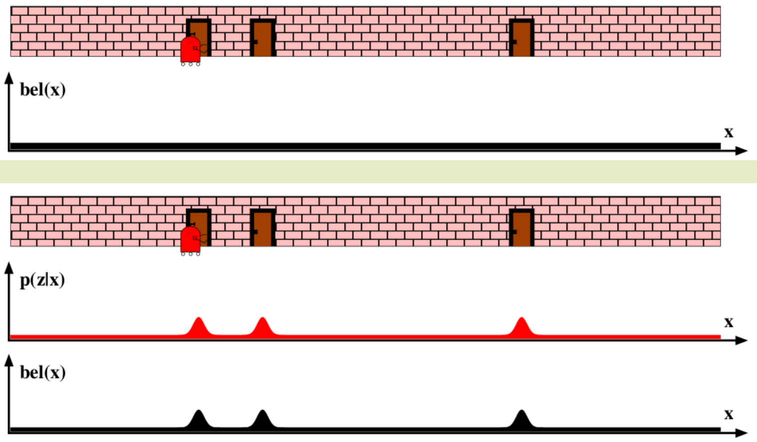

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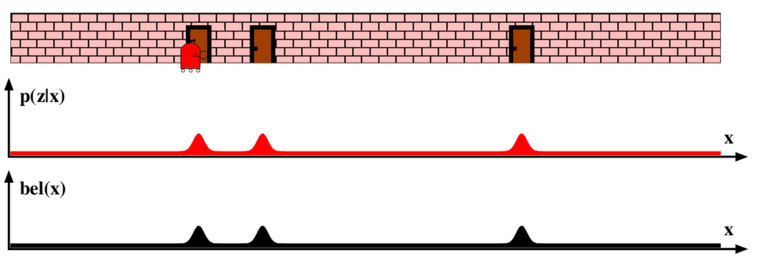

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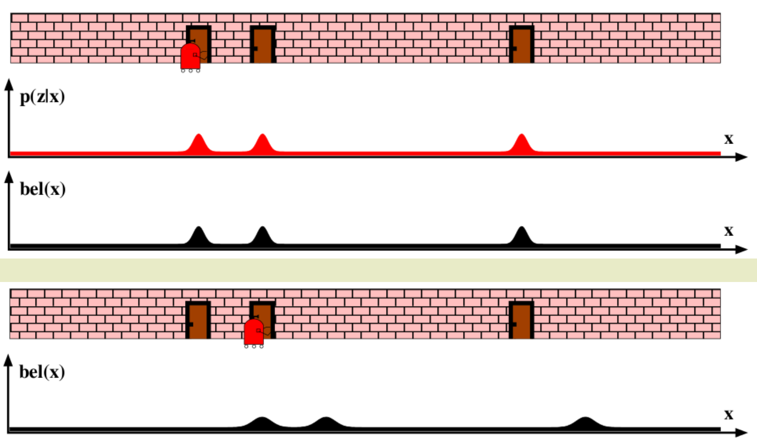

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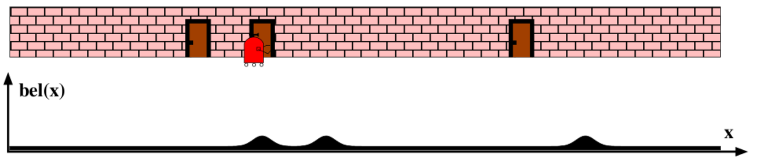

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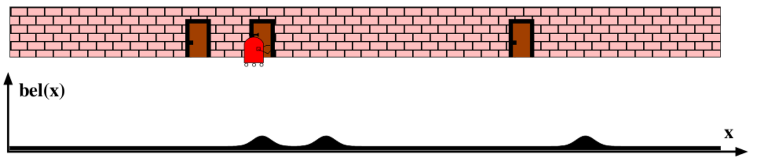

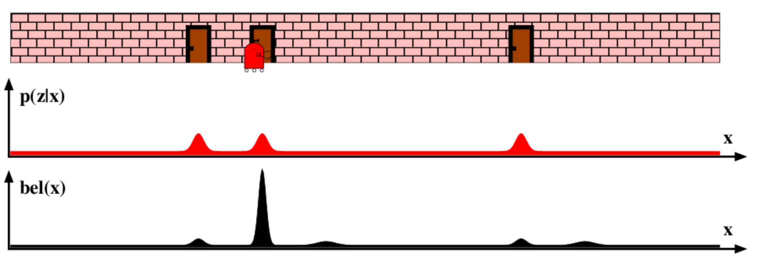

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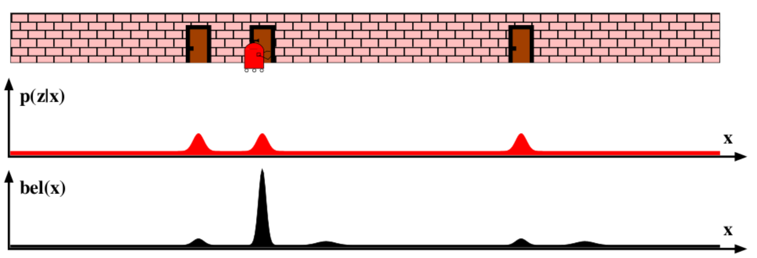

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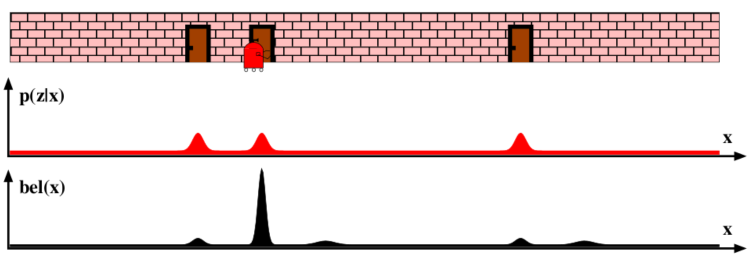

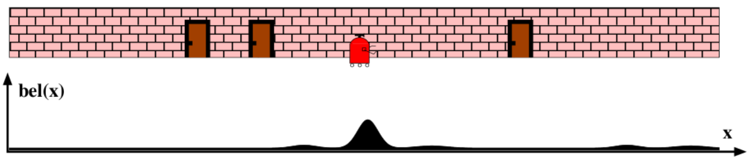

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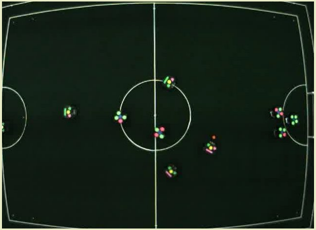

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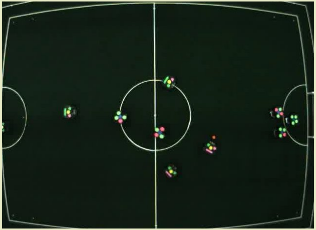

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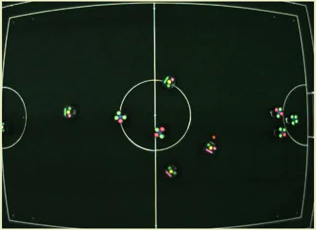

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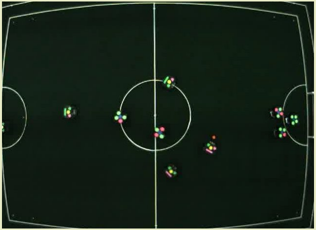

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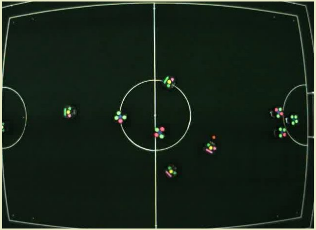

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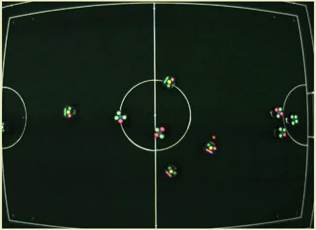

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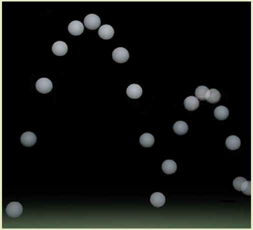

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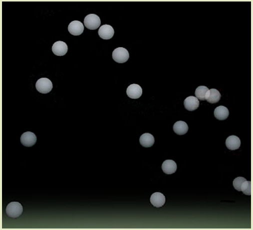

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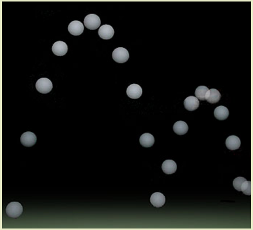

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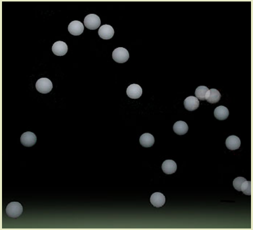

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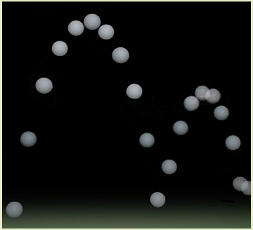

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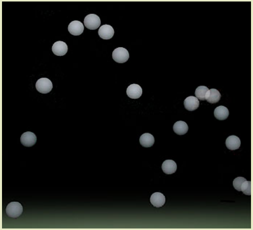

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- Sensor model: transformation from state to image, Gaussian distribution in the image
- Prediction: state transition model using physics

Bayes filter

- Previous slides have shown the principle of a *Bayes filter*.
- Why does this work exactly?
  - Probabilities
  - Bayes rule
  - Recursive Bayesian estimation

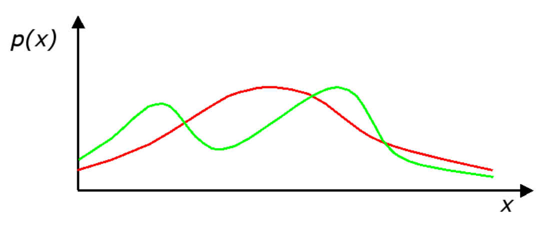
Source for the following slides: Thrun et al., Probabilistic Robotics; http://robots.stanford.edu/probabilistic-robotics/

Discrete random variables

- X denotes a random variable.
- X can take on a countable number of values in $\{x_1, x_2, ..., x_n\}$.
- $P(X = x_i)$ is the probability that X takes on value x_i.

Continuous random variables

- X takes on values in the continuum.
- $p(X = x)$ (or short $p(x)$) is a probability density function.
- Example: $Pr(x \in [a, b]) = \int_a^b p(x)dx$



Joint and Conditional Probabilities

- $P(X = x \ and \ Y = y) = P(x, y)$.
- If X and Y are independent then $P(x, y) = P(x)P(y)$.
- $P(x|y)$ is the probability of x given y.
- If X and Y are independent then $P(x|y) = P(x)$.

Law of total probability

- Discrete case:

$$\sum_x P(x) = 1$$

$$P(x) = \sum_y P(x, y)$$

$$P(x) = \sum_y P(x|y)P(y)$$

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- Continuous case:

$$\int p(x)dx = 1$$

$$p(x) = \int p(x, y)dy$$

$$p(x) = \int p(x|y)p(y)dy$$

Bayes rule

- $p(x|y)p(y) = p(x, y) = p(y|x)p(x)$

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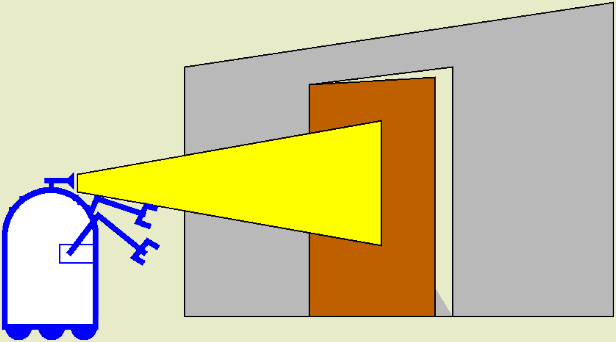
- $p(x|y) = \dfrac{p(y|x)p(x)}{p(y)} \overset{const\ \ y}{\propto} p(y|x)p(x)$
- Bayes rule with background knowledge:

  $$p(x|y, z) = \frac{p(y|x, z)p(x|z)}{p(y|z)}$$

Example for a simple measurement



- The robot obtains the measurement z.
- What is $P(open|z)$?

Diagnostic vs. causal reasoning

- $P(open|z)$ is diagnostic.
- $P(z|open)$ is causal.
- Often the causal knowledge is much easier to obtain (the *sensor models*).

Diagnostic vs. causal reasoning

- $P(open|z)$ is diagnostic.
- $P(z|open)$ is causal.
- Often the causal knowledge is much easier to obtain (the *sensor models*).
- The bayes rule allows us to use causal knowledge to get $P(open|z)$:

$$P(open|z) = \frac{P(z|open)P(open)}{P(z)}$$

Example

- $P(z|open) = 0.6 \qquad P(z|\neg open) = 0.3$
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- $P(open|z) = \dfrac{0.6 * 0.5}{0.6 * 0.5 + 0.3 * 0.5} = \dfrac{2}{3} \approx 0.67$
- The measurement z raises the probability that the door is open.

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$$P(x|u, x')$$

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  - Continuous: $p(x|u) = \int p(x|u, x')p(x')dx'$

Markov assumptions

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$$p(z_t|x_t,\ldots) = p(z_t|x_t)$$

Markov assumptions

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$$p(z_t|x_t, ...) = p(z_t|x_t)$$

- State x_t only depends on x_{t-1} and u_{t-1}:

$$p(x_t|u_{t-1}, x_{t-1}, ...) = p(x_t|u_{t-1}, x_{t-1})$$

Bayes filter

- Given:
  - Measurements $z_1, \ldots, z_t$ and action data/transition measurements $u_1, \ldots, u_t$.
  - Sensor model: $p(z|x)$.
  - Action model: $p(x|u, x')$.
  - Prior probability of the state $p(x)$.
- Wanted:
  - Belief of the state: $Bel(x_t) = p(x_t|z_t, u_{t-1}, \ldots, u_1, z_1)$

Recursive Bayesian estimation

$$Bel(x_t) = p(x_t | z_t, u_{t-1}, z_{t-1}, \ldots)$$

Recursive Bayesian estimation

$$\begin{aligned} Bel(x_t) &= p(x_t|z_t, u_{t-1}, z_{t-1}, \ldots) \\ \text{Bayes} \qquad &= \frac{p(z_t|x_t, u_{t-1}, z_{t-1}, \ldots)p(x_t|u_{t-1}, z_{t-1}, \ldots)}{p(z_t|u_{t-1}, z_{t-1}, \ldots)} \end{aligned}$$

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z_t \ \text{const.} \quad &= \eta p(z_t | x_t, u_{t-1}, z_{t-1}, \ldots) p(x_t | u_{t-1}, z_{t-1}, \ldots)
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&= \eta p(z_t|x_t) \int p(x_t|u_{t-1}, x_{t-1})Bel(x_{t-1})
\end{aligned}
$$

Bayes filter implementations

$$Bel(x_t) = \eta p(z_t|x_t) \int p(x_t|u_{t-1}, x_{t-1}) Bel(x_{t-1})$$

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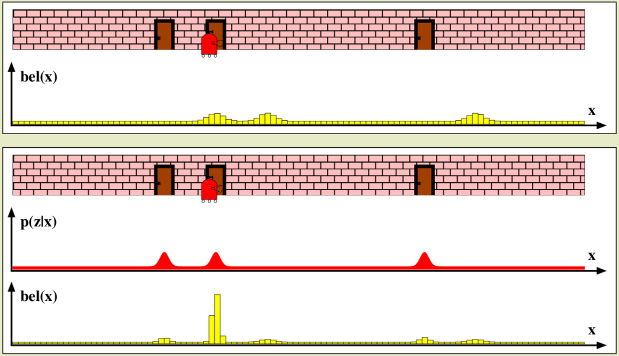
$$Bel(x_t) = \eta p(z_t|x_t) \int p(x_t|u_{t-1}, x_{t-1}) Bel(x_{t-1})$$

Some methods based on this equation:

- Grid-based estimator
- Kalman filter
- Particle filter

Grid-based estimator

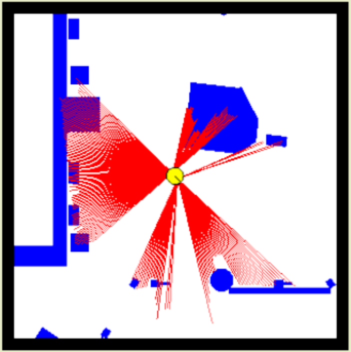
- Probability density function (belief) is represented using a discretized state space.
- Can be a simple grid with a constant step size.

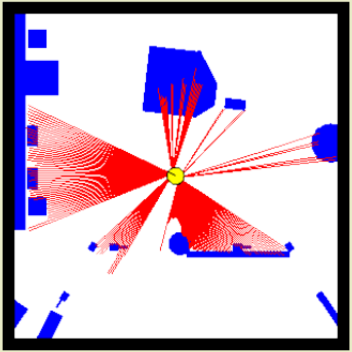


- Tree-based methods using e.g. octrees for more efficiency.

Grid-based estimator

- Can be useful e.g. for localizations using a grid-based environment map.





Kalman filter

- The belief is represented by multivariate normal distributions.
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- Extensions of the Kalman Filter for nonlinearity:
  - Extended Kalman Filter
  - Unscented Kalman Filter

Particle filter

- Belief represented by samples (particles).
- State estimation for non-Gaussian, nonlinear systems.

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- Belief represented by samples (particles).
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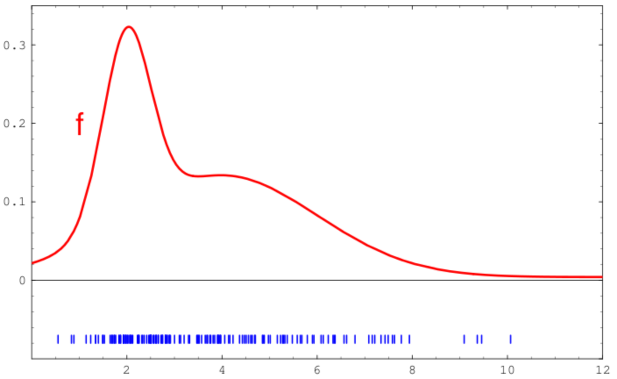
- Particles have weights.
- A high probability in a given region can be represented by
  - many particles.
  - few particles with higher weights.

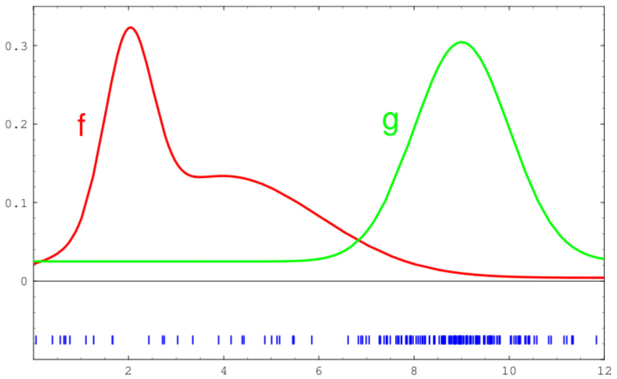
Importance sampling

- Suppose we want to approximate a target density f.



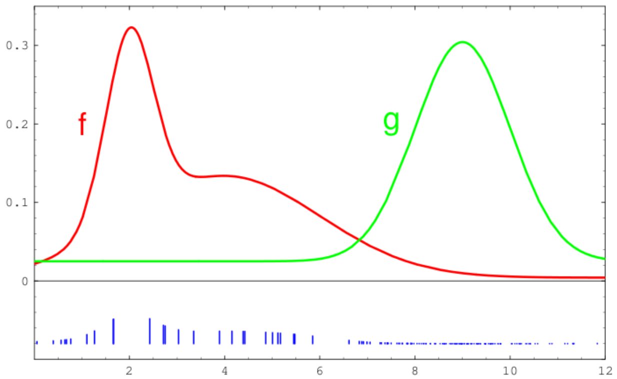
Importance sampling

- Assume we can only draw samples from a density g.



Importance sampling

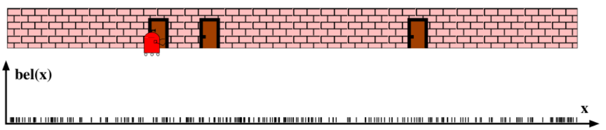
- The target density f can be approximated by attaching the weight $w = f(x)/g(x)$ to each sample x.



Example Monte Carlo localization

Sensor information (importance sampling)

$$Bel(x) \leftarrow \alpha p(z|x) Bel(x)$$



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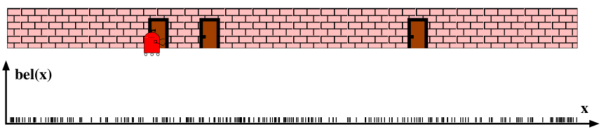

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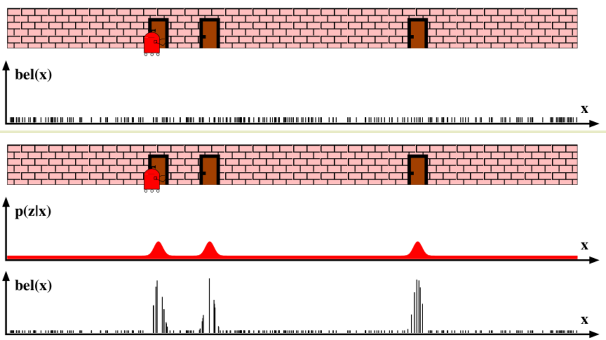
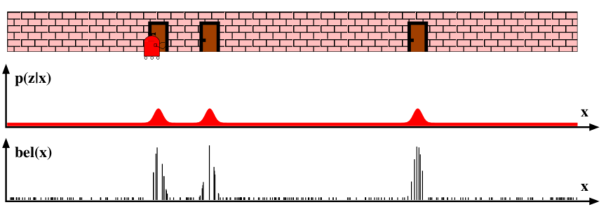


Example Monte Carlo localization

Robot motion (resampling and prediction)

$$Bel(x) \leftarrow \int p(x|u, x')Bel(x')dx'$$



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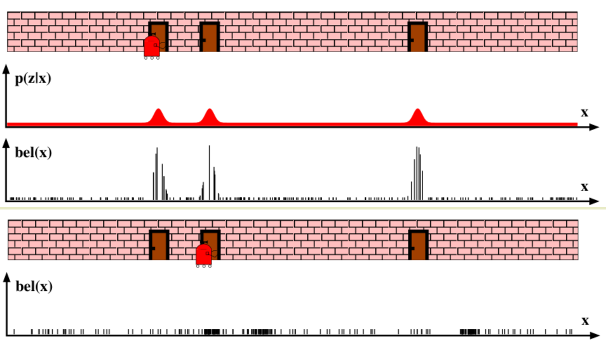

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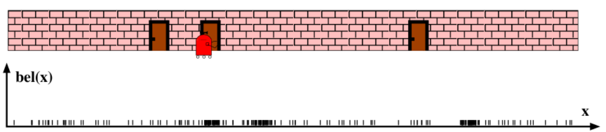

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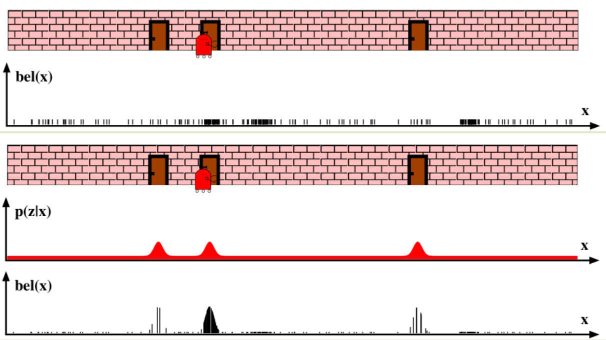

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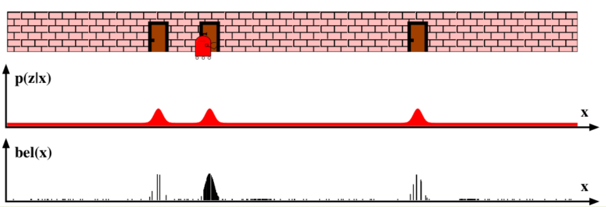

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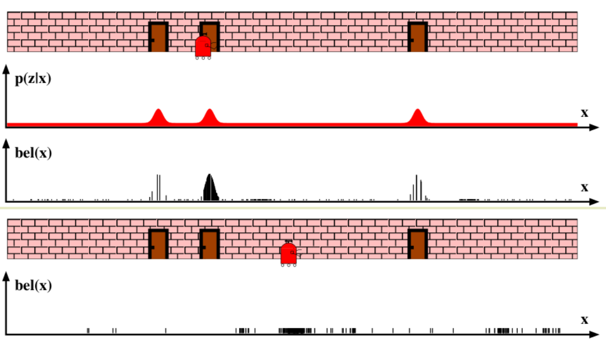


Particle filter steps

- State transition/prediction: Sample new particles using $p(x|u_{t-1}, x_{t-1})$.
  - In the context of localization: Move particles according to a motion model.
- Sensor update: Set particle weights using the likelihood $p(z|x)$.
- Resampling: Draw new samples from the old particles according to their weights.

Particle filter algorithm

procedure PARTICLE_FILTER(X_{t-1}, u_t, z_t)

    $\bar{X}_t = \emptyset, X_t = \emptyset$

    for $i = 1, \ldots, n$ **do** ▷ Generate new samples

        Sample x_t^i from $p(x_t | x_{t-1}^i, u_t)$

        $w_t^i = p(z_t | x_t^i)$ ▷ Compute importance weight

        $\bar{X}_t = \bar{X}_t + \langle x_t^i, w_t^i \rangle$ ▷ Update and insert normalization factor

    end for

    for $i = 1, \ldots, n$ **do** ▷ Resampling

        draw i with probability $\propto w_t^i$

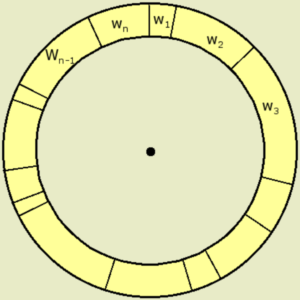
        add w_t^i to X_t

    end for

end procedure

Resampling

Resampling



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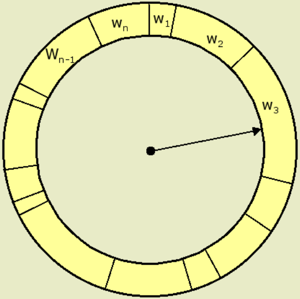

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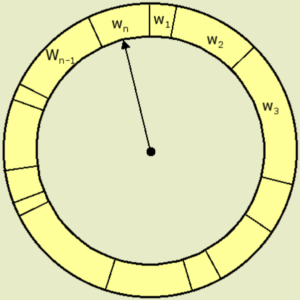

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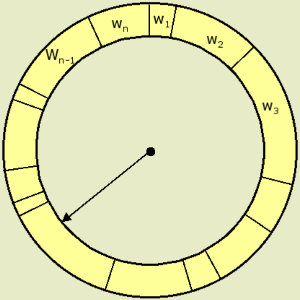

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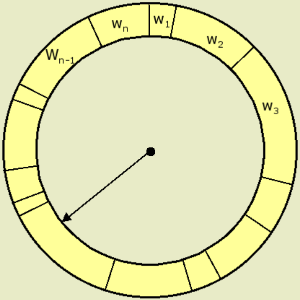
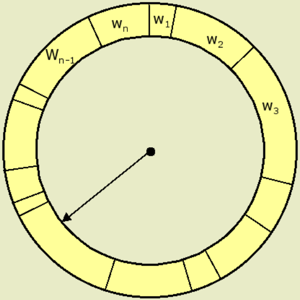
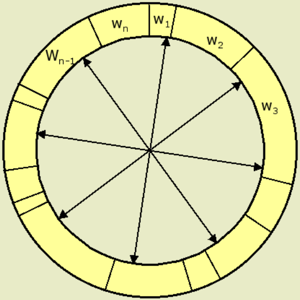


- Binary search, *n log n*
- High variance

Resampling





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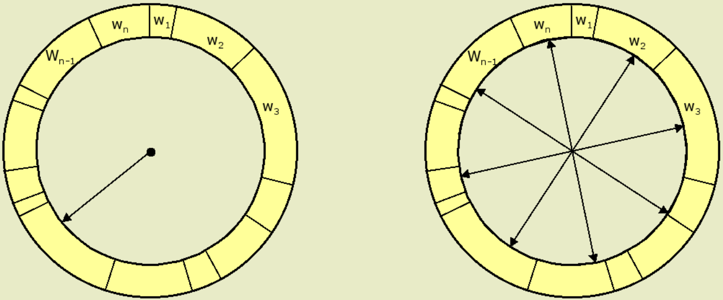

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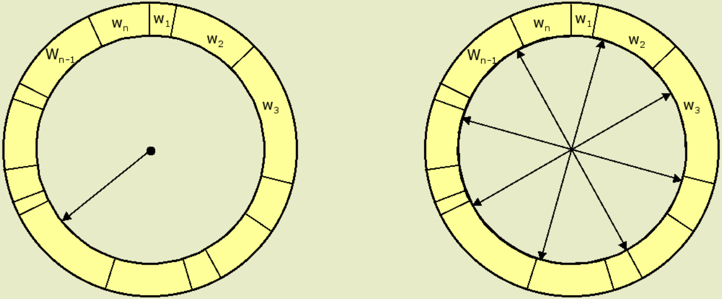

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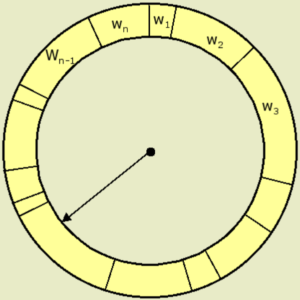
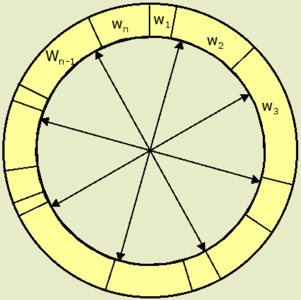
Resampling

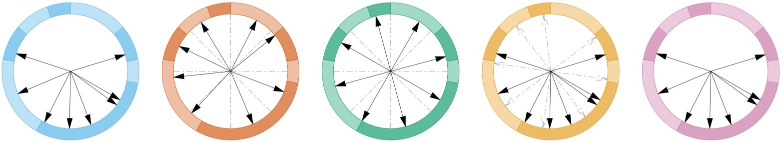




- Binary search, *n log n*
- High variance

Systematic resampling

- Stochastic universal sampling
- Linear time complexity
- Low variance



(a) Multinomial (b) Stratified (c) Systematic (d) Metropolis (e) Rejection

Source: Murray, Lawrence M., Anthony Lee, and Pierre E. Jacob. "Parallel resampling in the particle filter." arXiv preprint arXiv:1301.4019 (2013).

Resampling algorithm

procedure SYSTEMATIC_RESAMPLING(X_t, n)
  $X_t^{'} = \emptyset,\ c_1 = w^1$
  for $i = 2, \ldots, n$ **do** ▷ Generate cdf
    $c_i = c_{i-1} + w^i$
    $u_1 \sim U]0, n^{-1}]$ ▷ Initialize threshold
  end for
  for $j = 1, \ldots, n$ **do** ▷ Draw samples
    while $u_j > c_i$ **do** ▷ Skip until next threshold reached
      $i = i + 1$
      $S^{'} = S^{'} \bigcup \{\langle x^i, n^{-1} \rangle\}$ ▷ Insert
      $u_{j+1} = u_j + n$ ▷ Increment threshold
    end while
  end for
  Return $X_t^{'}$
end procedure ▷ Also called: **stochastic universal resampling**

Summary

- Particle filters are an implementation of a recursive Bayesian filter.
- Belief is represented by a set of weighted samples.

- Samples can approximate arbitrary probability distributions.
- Works for non-Gaussian, nonlinear systems.
- Relatively easy to implement.

- Depending on the state space a large number of particles might be needed.
- Re-sampling step: new particles are drawn with a probability proportional to the likelihood of the observation.

Problems

- Global localization problem (initial position).
- Robot kidnapping problem.

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- Global localization problem (initial position).
- Robot kidnapping problem.
- Augmented Monte Carlo Localization:
  - Inject new particles when the average weight decreases.
  - New random particles or particles based on current perception.

Acknowledgement

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