Exercises, Chapter 2

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Exercise 2.1-1: $f(n), g(n)$ asymptotically nonnegative. Show $\max(f,g) = \Theta(f + g)$.

Since f and g are asy. nonneg., so is $f+g$. For the upper bound, by the nonneg. of the functions:

$$f, g \leq f + g \;\; \Rightarrow \; \max(f,g) \leq f + g, \; \forall n > 0$$

This gives $\max(f,g) = O(f+g)$.

The lower bound is a bit trickier. The idea is that the maximum of two things can't be smaller than the average of those things:

$$1/2(f+g) \leq \max(f,g) \; \forall n > 0.$$

This is show by contradiction. Suppose $1/2(f+g) > \max(f,g)$. Without loss of generality, let f be the larger, $f > g$ for a certain n. For this n:

$$f + g > 2f \;\; \Rightarrow \;\; g > f$$

a contradiction. Therefore, we can take $n_o = 1$ and $c = 2$ in the definition of Ω. △

Exercise 2.1-2: $\forall a, b \in \mathbf{R}, \; b > 0, (n+a)^b = \Theta(n^b)$.

For the upper bound, let $n > a$ so that $n + a < 2n$. Hence $(n+a)^b < (2n)^b = c_u n^b$.

For the lower bound, let n be such that $n + a > (1/2)n$. Solving, this means $n > 2(-a)$. Hence $(n+a)^b > (1/2n)^b = c_l n^b$.

Since $(n+a)^b = O(n^b)$ and $(n+a)^b = \Omega(n^b)$, then $(n+a)^b = \Theta(n^b)$ △

Exercise 2.1-4: $2^{n+1} = O(2^n)$, because $2^{n+1} = 2(2^n)$.

$2^{2n} \neq O(2^n)$, since we can show $2^{2n} = \omega(2^n)$:

$$\lim_{n\to\infty} 2^n/4^n = (1/2)^n \to 0.$$

△

Exercise 2.1-5: $f = \Theta(g)$ if and only if $f = O(g)$ and $f = \Omega(g)$.

Suppose $f = \Theta(g)$,

$$f = \Theta(g) \;\; \Rightarrow \;\; \exists n_o, c_1, c_2 > 0 \text{ s.t. } 0 \leq c_1 f \leq g \leq c_2 f, \; \forall n \geq n_o,$$

Use c_1, n_o in the definition of $\Omega(g)$ and c_2, n_o in the definition of $O(g)$.

Suppose $f = O(g)$ and $f = \Omega(g)$.

$$f = O(g) \;\Rightarrow\; \exists n_1, c_1 > 0 \text{ s.t. } 0 \le f \le c_1 g,\ \forall n \ge n_1$$

and

$$f = \Omega(g) \;\Rightarrow\; \exists n_2, c_2 > 0 \text{ s.t. } 0 \le c_2 g \le f,\ \forall n \ge n_2$$

For $f = \Theta(g)$, set $n_o = \max(n_1, n_2)$ and the two constants to c_1 and c_2. △

EXERCISE 2.1-6: The best-case $T_{BC}(n)$ and worst-case $T_{WC}(n)$ run times are related to the run time $T(n)$ by:

$$T_{BC}(n) \le T(n) \le T_{WC}(n),\ \forall n > 0.$$

$$T_{BC} = \Omega(g) \;\Rightarrow\; c_{BC} g \le T_{BC}(n),\ \forall n \ge n_{BC}$$

$$T_{WC} = O(g) \;\Rightarrow\; T_{WC}(n) \le c_{WC} g,\ \forall n \ge n_{WC}$$

Let $n_o = \max(n_{BC}, n_{WC})$ and join the three inequalities. △

EXERCISE 2.1-7: The set $o(g) \cap \omega(g)$ is empty. Else, the f in the intersection satisfies,

$$\lim f/g \to 0 \text{ and } \lim g/f \to 0.$$

△

EXERCISE 2.2-1: If $f(n)$ and $g(n)$ are monotonically increasing, so are $(f+g)(n)$ and $f(g(n))$.

Suppose $n > m$, then by adding inequality $f(n) \ge f(m)$ with $g(n) \ge g(m)$ we have $(f+g)(n) \ge (f+g)(m)$, so $f+g$ is mono. increasing. Also, $n > m \;\Rightarrow\; g(n) \ge g(m) \;\Rightarrow\; f(g(n)) \ge f(g(m))$.

If also $f(n), g(n) \ge 0$ for all n, multiplying inequality $f(n) \ge f(m)$ and $g(n) \ge g(m)$ gives $f(n)g(n) \ge f(m)g(m)$ for $n > m$. So the product of the function s in mono.increasing. △

EXERCISE 2.2-2: $T(n) = n^{O(1)}$ if and only if $T(n) = O(n^k)$ for some $k > 0$.

$$T = n^{O(1)} \;\Rightarrow\; \exists g \in O(1) \text{ s.t. } T = n^{g(n)}.$$

But,

$$g \in O(1) \;\Rightarrow\; \exists\, k, n_o > 0 \text{ s.t. } g(n) \le k,\ \forall n \le n_o.$$

By monotonicity, $T = n^{g(n)} \le n^k$ for these n.

Conversely, since $T(n) \ge 0$ for large enough n,

$$T(n) = n^{\log_n T(n)}$$

with the special case that $n^{\log_n 0} = 0$. Since $T(n) = O(n^k)$, then,

$$\exists\, c, n_o > 0 \text{ s.t. } \log_n T(n) \le k \log_n n + \log_n c, \forall n \ge n_o,$$

and it can be seen that $\log_n T(n) = O(1)$. Note that in this case it is important to interpret $T(n) = n^{O(1)}$ as exact equality $T(n) = n^{g(n)}$ for some $g(n) = O(1)$, and this causes us extra headaches in the proof. $\triangle$

EXERCISE 2.2-3: For $a, b, n > 0$, $a^{\log_b n} = n^{\log_b a}$.

Take the log of the LHS,

$$\log_a a^{\log_b n} = \log_b n$$

and of the RHS,

$$\log_a n^{\log_b a} = \log_b a \log_a n = \log_b n$$

The strict monotonicity of the log function implies that $a^{\log_b n} = n^{\log_b a}$ as well. $\triangle$

EXERCISE 2.2-7: Prove by induction that $F_i = (\phi^i - \widehat{\phi}^i)/\sqrt{5}$ where

$$\phi, \widehat{\phi} = (1 \pm \sqrt{5})/2.$$

Calculate directly the basis case for F_0 and F_1.

For $F_i = F_{i-1} + F_{i-2}$ with $i > 1$, use the induction hypothesis, factor out a common power of ϕ and $\widehat{\phi}$ and use the identities $\phi^2 = \phi + 1$, $\widehat{\phi}^2 = \widehat{\phi} + 1$:

$$\begin{aligned}
F_i &= F_{i-1} + F_{i-2} \\
&= (1/\sqrt{5})(\phi^{i-1} - \widehat{\phi}^{i-1} + \phi^{i-2} - \widehat{\phi}^{i-2}) \\
&= (1/\sqrt{5})(\phi^{i-2}(\phi + 1) - \widehat{\phi}^{i-2}(\widehat{\phi} + 1)) \\
&= (1/\sqrt{5})(\phi^i - \widehat{\phi}^i)
\end{aligned}$$

$\triangle$

EXERCISE 2.2-8: For $i \geq 0$, $F_{i+2} \geq \phi^i$.

Calculate directly the basis case for $i = 0$ and $i = 1$.

Use induction for $i > 1$:

$$\begin{aligned}
F_{i+2} &= F_{i+1} + F_i \\
&\geq \phi^{i-1} + \phi^{i-2} \\
&= \phi^{i-2}(\phi + 1) \\
&= \phi^{i-2}\phi^2 = \phi^i
\end{aligned}$$

$\triangle$