Proofs for Algorithms, 1

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Theorem 1 *For $x \in \mathbf{R}$ and $a, b \in \mathbf{Z},\ a, b > 0$,*

$$\lfloor x/ab \rfloor = \lfloor \lfloor x/a \rfloor / b \rfloor$$

and

$$\lceil x/ab \rceil = \lceil \lceil x/a \rceil / b \rceil.$$

Proof: To motivate the proof, consider the plane $\mathbf{R} \times \mathbf{R}$ with horizontal lines drawn at $y = 1$, $y = b$ and $y = ab$ and rays drawn from the origin passing through the integer points on $y = 1$, that is, $(i, 1),\ i = 1, 2, \ldots$. These rays also pass through some integer points on $y = b$ and $y = ab$.

The set of all x on the line $y = ab$ such that $\lceil x/ab \rceil = k$ lies inside the cone describe by rays from the origin passing through $((k-1)ab, ab)$ and (kab, ab), including the rightmost ray but excluding the leftmost ray. The intersection of this half open cone with the line $y = b$ gives all x' such that $\lceil x'/b \rceil = k$.

The calculation $\lceil x/a \rceil$ follows the ray from the origin passing through (x, ab) to its intersection with the $y = b$ line and then moving right along $y = b$ to the next integer point. Following the (x, ab) ray leaves us within the half-open cone, and moving right to the next integer point does not leave the cone, since the ray through (kab, ab) which is the righthand boundary of the cone also passes through the integer point (kb, b).

We can follow this picture to state a formal proof:

$$\begin{array}{lclcll}
(k-1)ab & < & x & \leq & kab & \text{for some } k \in \mathbf{Z} \\
(k-1)a & < & x/b & \leq & ka & \text{use } b > 0 \\
(k-1)a & < & \lceil x/b \rceil & \leq & ka & \text{use } a \in \mathbf{Z} \\
(k-1) & < & \lceil x/b \rceil / a & \leq & k & \text{use } a > 0
\end{array}$$

hence $\lceil \lceil x/b \rceil / a \rceil = k$.

We did use that $a \in \mathbf{Z}$, and the following example shows that this is necessary. Let $a = b = \sqrt{2}$ and $x = 2$. Do the math. However, we did not need that $b \in \mathbf{Z}$, hence we have a slightly stronger result.

△

Theorem 2 *Let $a_1, \ldots, a_d \in \mathbf{R}$ and $a_d > 0$. Then,*

$$\sum_{i=0}^{d} a_i x^i = \Theta(x^d).$$

PROOF: For $x > 1$,

$$\sum_{i=0}^{d-1} a_i x^i \le \sum_{i=0}^{d-1} |a_i|\, x^i \le x^{d-1} K$$

where we have set

$$K = \sum_{i=0}^{d-1} |a_i|$$

for notational convenience. We likewise show,

$$\sum_{i=0}^{d-1} a_i x^k \ge -x^{d-1} K.$$

Hence,

$$a_d x^d - K x^{d-1} \le a_d x^d + \sum_{i=0}^{d-1} a_i x^i \le a_d x^d + K x^{d-1},$$

that is,

$$x^d (a_d - K/x) \le \sum_{i=0}^{d} a_i x^i \le x^d (a_d + K/x).$$

Taking n_o large enough so that $K/n_o \le a_d/2$, for all $x \ge n_o$,

$$(a_d/2) x^d \le \sum_{i=0}^{d} a_i x^i \le (3a_d/2) x^d.$$

Hence $\sum a_i x^i = \Theta(x^d)$. △

Theorem 3 *For all $d, \epsilon > 0$, $\log^d n = o(n^\epsilon)$*

PROOF: We first prove by induction the case $d \in \mathbf{Z}$. Basis $d = 1$. Apply L'Hospital's rule to the indeterminate form,

$$\lim_{n\to\infty} (\log n)/n^\epsilon = \lim_{n\to\infty} (1/n)/(\epsilon n^{\epsilon-1}) = \epsilon/n^\epsilon \to 0.$$

Hence for any $c > 0$ there is an n_o such that for $n \ge n_o$, $(\log n)/n^\epsilon < c$. That is, $\log n = o(n^\epsilon)$.
Applying L'Hospital's to the case of the general power,

$$\lim_{n\to\infty} (\log^d n)/n^\epsilon = \lim_{n\to\infty} d(\log^{d-1} n)(1/n)/(\epsilon n^{\epsilon-1}) = (d/\epsilon) \lim_{n\to\infty} (\log^{d-1} n)/n^\epsilon$$

Hence $\log^d n = o(n^\epsilon)$ if $\log^{d-1} n = o(n^\epsilon)$, which complete the induction step.
By the monotonicity in d of $\log^d n$, the result extends to all $d \in \mathbf{R}$, $d > 0$. △

Theorem 4 *For all $d > 0$ and $a > 1$, $n^d = o(a^n)$*

PROOF: The proof pattern is the same as the previous theorem. L'Hospital's applied the case $d = 1$ gives,

$$\lim_{n\to\infty} n/a^n = \lim_{n\to\infty} 1/(a^n \log a) = 0.$$

Again, use L'Hospital's to prove the induction step and then extend to all real $d \geq 1$ by monotonicity. △

Corollary 1 *For all $\epsilon > 0$, $n \log n = o(n^{1+\epsilon})$.*

PROOF: This follows from a general result, if $f_1(n) = O(g_1(n))$ and $f_2(n) = o(g_2(n))$, then $f_1(n)f_2(n) = o(g_1(n)g_2(n))$. Applying this by setting $f_1(n) = n$ and $f_2(n) = \log n$.
To show the general result, let c_1 and n_1 be such that,

$$f_1(n) < c_1\, g_1(n) \;\text{ for all } n > n_1$$

This is possible because $f_1(n) = O(g_1(n))$. Let $c > 0$ be chosen, and set $c_2 = c/c_1$. Since $f_2(n) = o(g_2(n))$, there exists an n_2 such that,

$$f_2(n) < c_2\, g_2(n) \;\text{ for all } n > n_2.$$

Multiplying the equalities (both are positive) and letting n_o be the maximum of n_1 and n_2,

$$f_1(n)f_2(n) < (c_1 g_1(n))(c_2 g_2(n)) = c\, g_1(n)g_2(n) \;\text{ for all } n > n_o.$$

△