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Test 1 Answers

FEBRURARY 24, 5:00–6:15

The four problems were graded as follows:

1. 5 points.
2. (a) 2 points.
   (b) 1 point.
   (c) 2 points.
3. (a) 2 points for L_i, 2 points with for P_i.
   (b) 1 point.
4. $\Rightarrow$ 3 points.
   $\Leftarrow$ 2 points.

1. [SIMPLEX METHOD]

   Solve the Following LP showing step-by-step the simplex method:

$$
\begin{array}{lccccccc}
\max & x_1 & + & 2x_2 & + & x_3 & & \\
\text{s.t.} & x_1 & + & x_2 & + & x_3 & \leq & 2 \\
 & x_1 & + & x_2 & & & \leq & 1 \\
 & x_1, & x_2, & x_3 & \geq & 0 & &
\end{array}
$$

Introduce slack variables:

$$
\begin{aligned}
x_4 &= 2 - x_1 + x_2 + x_3 \\
x_5 &= 1 - x_1 - x_2 \\
z &= x_1 + 2x_2 + x_3
\end{aligned}
$$

The initial basis is $\{x_4 = 2, x_5 = 1\}$. We pivot x_2 into the basis, and x_5 out:

$$
\begin{aligned}
x_2 &= 1 - x_1 - x + 3 \\
x_4 &= 1 - x_3 + x_5 \\
z &= 2 - x_1 + x_3 - 2x_5
\end{aligned}
$$

Now pivot x_3 in and x_4 out:

$$
\begin{aligned}
x_2 &= 1 - x_1 - x_5 \\
x_3 &= 1 - x_4 + x_5 \\
z &= 3 - x_1 - x_4 - 2x_5
\end{aligned}
$$

The coefficients of z are all negative. Hence the optimal solution is $x_1 = x_4 = x_5 = 0, x_2 = x_3 = 1$ and it has cost 3.

2. [DUALITY]

   (a) Give the Dual of the previous LP problem.
   (b) Find the optimal dual solution, using whatever method you wish.
   (c) Demonstrate the Complementary Slackness conditions for your optimal dual/primal solution pair. That is, what should be true and what is true for each of the 5 variable-inequality pairings.

The dual is:

$$\text{minimize } 2y_1 + y_2$$

subject to:

$$\begin{aligned} y_1 + y_2 &\geq 1 \\ y_1 + y_2 &\geq 2 \\ y_1 &\geq 1 \end{aligned}$$

and $y_1, y_2 \geq 0$. Two of the inequalities are redundant.

The solution $y_1 = y_2 = 1$ is feasible and has a cost equal to the optimal primal solution. Hence this is the optimal dual solution.

Here are the five complementary slackness conditions:

(a) $y_1 + y_2 > 1 \;\Rightarrow\; x_1 = 0.$

(b) $x_2 \neq 0 \;\Rightarrow\; y_1 + y_2 = 2.$

(c) $x_3 \neq 0 \;\Rightarrow\; y_1 = 1.$

(d) $y_1 \neq 0 \;\Rightarrow\; x_1 + x_2 + x_3 = 2.$

(e) $y_2 \neq 0 \;\Rightarrow\; x_1 + x_2 = 1.$

where the first three come from the primal variables, the last two come from the dual variables. The equalities are all satisfied.

3. [LU DECOMPOSITION]

(a) Use Gaussian Elimination with partial pivoting to decompose,

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 0 & 3 & 3 \end{bmatrix},$$

into the product,

$$L_3 P_3 L_2 P_2 L_1 P_1 A = U,$$

where L_i are column i eta-matrices, P_i are permutation matrices, and U is upper triangular with 1's down the diagonal.

(b) Use back substitution and your decomposition to find x_1, x_2, x_3 real numbers which satisfy,

$$\begin{array}{ccccccc} x_1 & + & x_2 & + & x_3 & = & 1 \\ x_1 & & & + & x_3 & = & 1/2 \\ & & x_2 & + & x_3 & = & 1/3 \end{array}$$

$$P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad L_1 = \begin{bmatrix} 1/2 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & -1/3 & 1 \end{bmatrix}$$

$$P_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad L_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Part (b) seeks x such that $Ax = [1, 1, 1]$ (a column vector of 1's). First calculate:

$$L_3 P_3 L_2 P_2 L_1 P_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = L_3 P_3 L_2 P_2 \begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix}$$

$$
\begin{aligned}
&= L_3 P_3 L_2 \begin{bmatrix} 1/2 \\ 1 \\ 1/2 \end{bmatrix} \\
&= L_3 P_3 \begin{bmatrix} 1/2 \\ 1/3 \\ 1/6 \end{bmatrix} \\
&= \begin{bmatrix} 1/2 \\ 1/3 \\ -1/6 \end{bmatrix}
\end{aligned}
$$

Then back substitute:

$$
Ux = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} x = \begin{bmatrix} 1/2 \\ 1/3 \\ -1/6 \end{bmatrix}
$$

giving,

$$
x = \begin{bmatrix} 2/3 \\ 1/2 \\ -1/6 \end{bmatrix}.
$$

4. [THEORY]

Prove that the product AB of two square matrices is nonsingular if and only if both A and B are nonsingular.

Suppose A and B are nonsingular. By Theorem 6.2, a matrix is nonsingular if and only if there exists an inverse. In our case, there exist A^{-1} and B^{-1} such that,

$$AA^{-1} = A^{-1}A = B^{-1}B = B^{-1}B = I.$$

Then,

$$AB(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I,$$

and,

$$(B^{-1}A^{-1})AB = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I.$$

So AB has an inverse, namely $B^{-1}A^{-1}$, and is therefore nonsingular.

Now suppose that AB is nonsingular. By Theorem 6.1, a square matrix has two possibilities only: it is nonsingular and each equation $Ax = b$ has a unique solution for x, or it is singular and the equation $Ax = b$ has either no or an infinity of solutions, depending on the b. It is enough to show that $Ax = 0$ and $Bx = 0$ have unique solutions. Because $Ax = 0$ has a solution, $x = 0$, if it is singular it must have an infinity of solutions $Ax = 0$. If it has only one solution, than it must be nonsingular. The same goes for B.

Suppose $Bx_1 = Bx_2 = 0$. Then $ABx_1 = ABx_2 = 0$ so $x_1 = x_2 = 0$. Thus B is nonsingular. If $Ax_1 = Ax_2 = 0$, then since B is non-singular, there are unique solutions to $By_i = x_i$, with $i = 1, 2$. Hence $ABy_i = 0$ and thus $y_1 = y_2$, and $x_1 = By_1 = By_2 = x_2$. So A is nonsingular.