Solution Set 1

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1. Problem 1.1.1: Construct a proper infinite ascending chain of partial functions mapping naturals to naturals. One such family could be,

$$\psi_i(n) = \begin{cases} n & n \leq i \\ \perp & \text{else} \end{cases}$$

   Note that for $i < j$, $DD(\psi_i) \subset DD(\psi_j)$, $DD(\psi_i) \neq DD(\psi_j)$ and for all $n \in DD(\psi_i),\ \psi_i(n) = \psi_j(n)$.

2. Problem 1.1.2: The function $\Theta(a,b) = a/b$ is not total because it is not defined when $b = 0$. Its range is $\mathbf{Q}$, the set of rationals.

   - Surjectivity: its range covers all rational numbers. For any $r \in \mathbf{Q}$, $\Theta(r,1) = r$.
   - Into: its range is restricted to only the rational numbers. Given any rationals a and b, we can write them as $a = a_n/a_d$ and $b = b_n/b_d$ with a_n, a_d, b_n and b_d integers. Then $\Theta(a,b) = a_n b_d / a_d b_n$, a rational number.

3. Problem 1.1.3: The range of $f(n) = 1 + 3 + \ldots + (2n-1)$ is the set of all squares,

$$\{\, f(n) = n^2 \mid n = 1, 2, \ldots \}$$

   Proof is by induction.

   - Basis, $f(1) = 1 = 1^2$.
   - Induction. Suppose $f(n) = n^2$ for any $n \leq N$. Then,

$$\begin{aligned} f(N+1) &= 1 + 2 + \ldots + 2(N+1) - 1 \\ &= f(N) + 2(N+1) - 1 \\ &= N^2 + 2N + 1 = (N+1)^2. \end{aligned}$$

   Therefore, by induction, $f(n) = n^2$ for all n.

4. Problem 1.3.1(e): The set of partial functions mapping naturals to naturals with finite domains is countable. Let this set be called S. Here is a bijection $\psi : S \to \mathbf{N}$,

$$\psi(f) = \prod_{i \in DD(f)} {pr_i}^{f(i)}$$

where

- $DD(f)$ is the finite set of integers at which f is defined,
- pr_i is the i-th prime, $pr_0 = 2,\ pr_1 = 3\ pr_2 = 5$, etc.,
- $f(i)$ is the value of f at i,
- Π is "product" in the same sense one uses Σ for sum.

For example, the function,

$$f(x) = \begin{cases} 4 & x = 0 \\ 7 & x = 3 \\ 2 & x = 4 \end{cases}$$

maps to $\psi(f) = 2^4 7^7 11^2$. Note that any integer can be factored to reveal the unique function it represents, and any function in S is represented by some integer. Hence this is a bijection.

5. Problem 1.3.2: The set of all total functions $\mathbf{N} \to \{0, 1\}$ is uncountable. Suppose, by way of gaining a contradiction, this collection was countable. Here is a counting:

$$f_0, f_1, f_2, \ldots.$$

Make a table of $f_i(j)$,

	0	1	2	$\cdots$
f_0	1	0	1	$\cdots$
f_1	0	0		
$\vdots$		$\cdots$	$f_i(j)$	$\cdots$

Define,

$$g(i) = \begin{cases} 1 & \text{if } f_i(i) = 0 \\ 0 & \text{if } f_i(i) = 1 \end{cases}$$

The function g is total from naturals to $\{0, 1\}$ and should therefore be in the table. But if $g = f_j$ for some j then the value of $g(j) = f_j(j)$ is 0 if and only if its value is 1. Oops.

The set of subsets of $\mathbf{N}$ is uncountable. This is reducible to the previous proof by considering the "characteristic function" of a subset. For $S \subseteq \mathbf{N}$ a subset of the naturals, the characteristic function of S is:

$$\chi_S(n) = \begin{cases} 1 & \text{if } n \in S \\ 0 & \text{if } n \notin S \end{cases}$$

Therefore each total function from naturals to the set $\{0, 1\}$ is a subset of the naturals and vice-a-versa.

Finally, the set of partial functions from naturals to naturals with finite range includes the total functions from naturals to the set $\{0, 1\}$ and is therefore bigger. If some map surjects this larger set then a restriction of this map surjects the subset which we know to be uncountable. Hence, no surjection from the naturals to the set of partial functions with finite range can exist.

6. Problem 1.3.3: From problem 1.3.2, the infinity of total mappings from naturals to naturals is uncountable. However the infinity of programs is countable. There just aren't enough programs to go around.

7. Problem 1.3.4: See discussion on page 77 of the book.

8. Problem 1.3.5: It is decidable if a program on a computer of finite size halts. Considering all the computer's memory as well as instruction pointer register, stack pointer register, etc., as the machines "state", a finite computer is a finite state machine. Let N be the total number of states in this machine. Either the computer halts in N steps or else it has visited a state more than once and we can conclude that the computer is stuck in an infinite loop. The diagonalization argument applied to a finite machine only shows that there are functions not computable on a machine of fixed size.