Solution Set 4

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1. Problem 3.2.2: If $\mathbf{P}_e$ is a while-program computing $\Phi : N^2 \to N$, show $\mathbf{P}_e(e, a) = \mathbf{P}_e(a, 0)$ for all a.

   If $\mathbf{P}_e(e, a)$ halts then $\Phi(e, a)$ is defined and therefore $\mathbf{P}_e(a)$ halts. Conversely, if $\mathbf{P}_e(a)$ halts, then $\Phi(e, a)$ is defined so $\mathbf{P}_e(e, a)$ halts. Therefore:
   $$\mathbf{P}_e(e, a) = \Phi(e, a) = \mathbf{P}_e(a),$$
   where equality is understood in the extended sense where, in addition to the standard equality of naturals, "non-halting" equals "not-defined".

   By Definition 6, Section 2.3, $\mathbf{P}_e(a) = \mathbf{P}_e(a, 0)$. This completes the proof.

2. Problem 3.2.3: Let $f : N \to N$ be a computable bijection. Enumerate all while-programs by:
   $$\mathbf{P}_{f(0)}, \mathbf{P}_{f(1)}, \ldots .$$
   Prove the existence of a universal function Ψ for this enumeration and show $\Psi = \mathbf{P}_{f(n)}$ for some n.

   The principal result of Chapter 3 is that the universal function $\Phi(x, y)$ is computable. But then so is,
   $$\Psi(a, b) = \Phi(f(a), b),$$
   which is the universal function for the new enumeration. So $\Psi(a, b)$ is given by some arity two program $\mathbf{P}_m$. The surjectivity of f assures the existence of an n such that $m = f(n)$. Therefore, the statement remains true if f is total and onto but not if it is total and one-to-one but not onto.

3. Problem 3.2.4: Let $\Phi : N^2 \to N$ be the universal function. Show that $\theta(x) = \Phi(x, x)$ cannot be extended to a total computable function.

   Following the hint, let $\theta' \geq \theta$ be total and computable. Let
   $$\theta''(x) = \theta'(x) + 1.$$

It is clear that θ'' is computable if θ' is, so $\theta'' = \mathbf{P}_j$. So,

$$\theta'(j) + 1 = \theta''(j) = \mathbf{P}_j(j) = \Phi(j, j) = \theta(j) = \theta'(j).$$

Note that each of these equalities is a true equality: on the assumption that θ' is total, each function is defined at j and the j-th while-program halts. We use this in the assertion $\theta'(j) = \theta(j)$ which is only true if θ is defined at j. But $\theta'(j) = \theta'(j) + 1$ is a contradiction.

In order to understand this proof better. Suppose $\theta' \geq \theta$ was a computable function but perhaps not total. We still can define a computable θ'' as before and the index j still exists. The equation,

$$\theta'(j) + 1 = \theta'(j),$$

which has no solutions over the naturals, does have a solution if we allow $\theta'(j) = \perp$ — a non-terminating computation followed by an application of the successor function is still a non-terminating computation. Therefore θ' is not total.

4. Problem 3.2.5: Show the existence of a function ϕ_i for which no algorithm can decide if $\phi_i(x)$ is defined for an arbitrary $x \in N$.

   Let $\phi_i(x) = \Phi(x, x)$. This function is computable. Note that the question whether $\phi_i(x)$ is defined on x is equivalent to whether the x-th while-program halts on x, which is not computable.