Solution Set 5

DATE: 27 OCTOBER, 1992

A midterm solution set will not be made available.

1. Problem 4.1.3: Let α and β be unary computable functions such that $A = \mathrm{DOM}(\alpha), B = \mathrm{DOM}(\beta)$, and $A \cap B = \emptyset$.

   Does there necessarily exist a computable function γ such that $\gamma(A) = \{0\}$ and $\gamma(B) = \{1\}$? Answer: yes. Proof: dovetail the two computations.

   Does there necessarily exist a total computable function f such that $f(A) = \{0\}$ and $f(B) = \{1\}$. Answer: no. Proof: Consider the insidious function,

   $$\psi(n) = \overline{\varphi_n(n)},$$

   meaning that, where $\varphi_n(n)$ is defined, $\psi(n)$ takes the opposite value. This is computable. The functions α and β can be derived from ψ as having proper domains for the problem statement. A total, computable extension f of ψ would have index j and $\varphi_j(j)$ must halt (because f is total). So,

   $$f(j) = \psi(j) = \overline{\varphi_j(j)} = \overline{f(j)}.$$

   A contradiction.

2. Problem 4.1.4: Discuss the computational status of each of the unary functions f_1, f_2, and f_3 below — according to whether or not it is the case that for all $i, j, k \in \mathbf{N} - \{0\}$ and all $n > 2 : i^m + j^n \neq k^n$ (Fermat's last theorem).

   $$f_1(n) = \begin{cases} 1 & \exists i, j, k \neq 0 : i^n + j^n = k^n \\ \perp & \text{else} \end{cases}$$

   This is computable whether or not Fermat's last theorem holds: just try all i, j, k.

   $$\begin{aligned} f_2(n) &= \begin{cases} \perp & \exists i, j, k : \ldots \\ 0 & \text{else} \end{cases} \\ f_3(n) &= \begin{cases} 1 & \exists i, j, k : \ldots \\ 0 & \text{else} \end{cases} \end{aligned}$$

If Fermat's is true, then these are computable. In fact, $f_2(n)$ is the sequence $\{\, 0, \perp, \perp, 0, 0, \ldots\}$ and f_3 is similarly definable. If Fermat's is not true, then for some n perhaps i, j, k exist which satisfy the equation $i^n + j^n = k^n$. Then f_3 computable implies f_2 computable, obviously. However, f_2 computable and f_1 computable implies f_3 computable, by dovetailing computations. So the computability of f_2 and f_3 are equivalent and reduces to the question of the decidability of the set;

$$\{\, n \in \mathbf{N} \mid \exists i, j, k \in \mathbf{N} - \{0\} \text{ such that } i^n + j^n = k^n \,\}$$

I do not believe the decidability of this set has been ascertained.