Least Fixed Points

DATE: 13 OCTOBER, 1992

Some non-examples of the least-fixed-point theorem.

Theorem 1 (Least Fixed Point) *Let A have a partial-order with least element $\perp$ and such that any ascending chain as a least upper bound. Let Θ be an endomorphism of A which is continuous and total. Then,*

$$Y(\Theta) = \bigvee_{n \geq 0} \Theta^n(\perp)$$

exists and is the unique least fixed point of Θ.

1. Take A the set of opens of a topological space except for the empty set. The A fails the hypothesis of the theorem only it that it lacks a least element. Let Θ be the identity, it is continuous and total. Yet is has no least fixed point.

2. Take A to be the set of all subsets of a set except the empty set and Θ the identity. The failure here is that A has no least element. All singletons are least fixed points.

3. Let A be the reals in the interval $[0, 1]$ with usual size ordering. It is an ω-cpo with $\perp = 0$. Let Θ be the discontinuous map,

$$\Theta(x) = \begin{cases} 1/4 + x/2 & x \in [0, 1/2) \\ 1/2 + x/2 & x \in [1/2, 1] \end{cases}$$

Then $\Theta(1) = 1$ is the least-fixed-point although the sup of $\Theta^n(\perp)$ is $1/2$.