Midterm

October 20, 1992. 5:30–8:00 PM

There are seven problems for a total of 100 points. Good luck.

Name: ______________________________

Problem	Credit
1	
2	
3	
4	
5	
6	
7	
Total	

1. (10 Points.)

   Are there more numbers than even numbers? To resolve this, either find a bijection between the naturals $\{0, 1, 2, \ldots\}$ and the even, positive numbers $\{0, 2, 4, \ldots\}$ or prove that none exists. What does this mean for the cardinality of the two sets: the set of naturals and the set of even naturals?

2. (10 Points.)

   Consider the following function:

$$f(n) = \begin{cases} n/3 & \text{if } n \text{ is divisible by 3,} \\ \perp & \text{else.} \end{cases}$$

   Give two different *total* functions g_1 and g_2 defined $N \to N$ which are extensions of f.

3. (20 Points.)

   For each of the following mark "u" if the set is uncountable, "c" if it is countable.

   - ☐ The set of all real numbers.
   - ☐ The set of all integers, both positive and negative.
   - ☐ Fix a real $\epsilon > 0$, positive but no matter how small. The set of all reals in the interval $[1, 1 + \epsilon]$.
   - ☐ The set of all subsets of the natural numbers.
   - ☐ The set of all *finite* subset of the natural numbers.

4. (15 Points.)

   Prove: It is undecidable whether an arbitrary while-program halts on some input value. I.e., the following function is not computable:

$$f(i) = \begin{cases} 1, & \text{if } \mathrm{DOM}(\varphi_i) \neq \emptyset; \\ 0, & \text{otherwise,} \end{cases}$$

   where φ_i is the i-th computable unary function in our standard enumeration.

5. (15 Points.)

   Write a while-program which computes the maximum of two numbers. I.e., the program moves to X1 the larger of the values in X1 and X2:

   $$X1 := \max(X1, X2)$$

   Use no macros!

6. (15 Points.)

   We define a new binary operation on functions, called *meet*. Given two partial functions f and g, the meet of f and g, $f \wedge g$, is the largest partial function h such that $h \leq f$ and $h \leq g$. That is, if h' is another partial function such that $h' \leq f$ and $h' \leq g$, then $h' \leq h$. In fact,

$$(f \wedge g)(i) = \begin{cases} f(i) \text{ or } g(i) & \text{when both are defined and equal,} \\ \perp & \text{elsewise.} \end{cases}$$

   Show that if f and g are computable, then $f \wedge g$ is computable.

7. (15 Points.)

   Use Church's Thesis to give an informal but rigorous argument that if a set of numbers cannot be effectively generated, then it is undecidable whether an arbitrary number is in the set.