Lecture 3.26. Hermitian, unitary and normal matrices

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March 10, 2023

In the previous lecture we considered matrices with non-negative entries. They frequently appear in applications to probability, statistics and social sciences, as it is demonstrated by examples in the previous lecture and section 5.3 of the textbook.

Now we consider important classes of matrices which are relevant to physics and engineering. I recall them.

We consider operators in a complex vector space with an Hermitian product. For example, $\mathbf{C}^n$ with the standard Hermitian product

$$(x, y) = x^*y = \overline{x_1}y_1 + \ldots + \overline{x_n}y_n.$$

I recall that "Hermitian transpose" of A is denoted by A^* and is obtained by transposing A and complex conjugating all entries. So for a real matrix $A^* = A^T$.

A matrix A is called *Hermitian* if

$$A^* = A.$$

Real Hermitian is the same as symmetric. A matrix U is called *unitary* if

$$U^*U = I.$$

So a real unitary matrix is the same as orthogonal. Examples:

$$A = \left(\begin{array}{cc} 2 & 1+i \\ 1-i & 1 \end{array} \right), \quad B = \left(\begin{array}{cc} a & -\overline{b} \\ b & \overline{a} \end{array} \right).$$

Matrix A is Hermitian, while B is unitary if and only if $|a|^2 + |b|^2 = 1$.

For these types of matrices we have the following important theorems.

Spectral theorem for Hermitian matrices. *For an Hermitian matrix,*
(i) all eigenvalues are real,
(ii) eigenvectors corresponding to distinct eigenvalues are orthogonal,
(iii) there is an **orthonormal** *basis consisting of eigenvectors.*

Spectral theorem for unitary matrices. *For a unitary matrix,*
(i) all eigenvalues have absolute value 1*,*
(ii) eigenvectors corresponding to distinct eigenvalues are orthogonal,
(iii) there is an **orthonormal** *basis consisting of eigenvectors.*

So Hermitian and unitary matrices are always diagonalizable (though some eigenvalues can be equal). For example, the unit matrix is both Hermitian and unitary. I recall that eigenvectors of *any* matrix corresponding to distinct eigenvalues are linearly independent. For Hermitian and unitary matrices we have a stronger property (ii).

Let me prove statements (i) of both theorems.

Suppose A is Hermitian, that is $A^* = A$. Let λ be an eigenvalue. This means that there exists a vector $\mathbf{v} \neq 0$ such that

$$A\mathbf{v} = \lambda\mathbf{v}.$$

Multiply this on $\mathbf{v}$ from the left. We obtain

$$\mathbf{v}^* A\mathbf{v} = \mathbf{v}^*\lambda\mathbf{v} = \lambda\mathbf{v}^*\mathbf{v} = \lambda\|v\|^2.$$

Now apply * operation to this equality:

$$\mathbf{v}^* A^*\mathbf{v} = \overline{\lambda}\|\mathbf{v}\|^2.$$

The left hand sides are equal since $A^* = A$, so the right hand sides have to be equal. But $\|\mathbf{v}\|^2 \neq 0$ (since eigenvector is not 0 by definition), so $\lambda = \overline{\lambda}$ that is λ is real.

Now suppose that A is unitary, that is $A^*A = I$. Let λ be an eigenvalue. This means that there exists $\mathbf{v} \neq 0$ such that

$$A\mathbf{v} = \lambda\mathbf{v}.$$

Then also

$$\mathbf{v}^* A^* = \overline{\lambda}\mathbf{v}^*$$

Let us multiply these equalities:

$$\mathbf{v}^* A^* A \mathbf{v} = \overline{\lambda} \mathbf{v}^* \lambda \mathbf{v} = |\lambda|^2 \|\mathbf{v}\|^2.$$

But since $A^* A = I$, the left hand side is $\|\mathbf{v}\|^2$. Since $\|\mathbf{v}\|^2 \neq 0$, we must have $|\lambda| = 1$.

So we proved (i) of both theorems. Since statements (ii) and (iii) are the same, it is reasonable to find a bigger class of matrices for which these two statements are true, and which contains both Hermitian and unitary matrices. This class consists of *normal matrices*.

A matrix is called *normal* if it satisfies

$$A^* A = A A^*.$$

Evidently Hermitian and unitary matrices are normal.

Exercise: give an example of a matrix which is normal but neither Hermitian nor unitary. Hint: an appropriate diagonal matrix will do the job.

Spectral theorem for normal matrices. *A matrix is normal is and only if there is an orthogonal basis of* $\mathbf{C}^n$ *consisting of eigenvectors.*

So normal matrices is the largest class for which statements (ii) and (iii) are true.

You can read the proof of this theorem in the handout “Spectral theorems for Hermitian and unitary matrices”.

Let us see what our theorems imply for real matrices. Suppose that A is real and symmetric. Then it is also Hermitian, so all eigenvalues are real. Then eigenvectors can be also chosen real, since they are solutions of linear equations with real coefficients, therefore

$$A = B \Lambda B^{-1},$$

where Λ is a real diagonal matrix and B is orthogonal (a real unitary matrix is orthogonal).

Now suppose that we have an orthogonal matrix Q. The eigenvalues are no longer guaranteed to be real, so in general, one cannot diagonalize Q using only real matrices.

Connection with exponentials.

Hermitian or real symmetric matrices are easy to understand: both classes are real vector spaces (a linear combination of Hermitian matrices with real coefficients is Hermitian, and same for real symmetric matrices).

Unitary (or orthogonal) matrices are more difficult.

Example: describe all 2×2 unitary matrices with determinant 1.

Let our matrix be

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where a, b, c, d are complex numbers. Let us try to write conditions on a, b, c, d which ensure that A is unitary. By definition this means that $AA^* = I$. Since $\det A = 1$,

$$A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

so we must have $a = \overline{d},\ c = -\overline{b}$, so

$$A = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}, \quad |a|^2 + |b|^2 = 1, \quad a, b \in \mathbf{C}.$$

This is a general form of a 2×2 unitary matrix with determinant 1.

In particular, when a, b are real, we obtain the general form of a 2×2 orthogonal matrix with determinant 1.

$$A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \quad |a|^2 + |b|^2 = 1, \quad a, b \in \mathbf{R}.$$

With matrices of larger size, it is more difficult to describe all unitary (or orthogonal) matrices.

Let H be a Hermitian matrix, that is $H^* = H$. I claim that the exponential of $U = \exp(iH)$ is unitary. Indeed,

$$U^* = \exp(-iH^*) = \exp(-iH) = U^{-1}.$$

Conversely, every unitary matrix U is an exponential of iH for some Hermitian H.

Indeed by the spectral theorem, for a unitary matrix U one can find a unitary matrix B such that

$$U = B\Lambda B^{-1}, \quad \Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n),$$

where $|\lambda_j| = 1$ for all j. Therefore we can write $\lambda_j = \exp(i\theta_j)$ with some real θ_j, and thus

$$U = Be^{i\Theta}B^{-1} = e^{iB\Theta B^{-1}}, \quad \text{where} \quad \Theta = \mathrm{diag}(\theta_1, \ldots, \theta_n).$$

Now the matrix $H = B\Theta B^{-1}$ is unitary, because

$$H^* = (B\Theta B^{-1})^* = (B^{-1})^*\Theta B^* = B\Theta B^{-1} = H,$$

since Θ is real and $B^* = B^{-1}$.

Thus *unitary matrices are exactly of the form e^{iA}, where A is Hermitian.*

Now we discuss a similar representation for orthogonal matrices. Let A be a real skew-symmetric matrix, that is $A^T = A^* = -A$. Then $-iA$ is Hermitian:

$$(-iA)^* = iA^* = iA^T = -iA.$$

So $e^A = e^{i(-iA)}$ is unitary, and since A is real, e^A is also real, thus e^A is orthogonal.

However we will not obtain *all* orthogonal matrices in this way. Indeed, for a skew symmetric matrix A, all main diagonal elements are zeros, so the trace is zero, and

$$\det e^A = e^{\mathrm{tr}\, A} = e^0 = 1.$$

One can show that the formula

$$e^A, \quad \text{where} \quad A \quad \text{is} \quad \text{skew symmetric}$$

actually represents all orthogonal matrices with determinant 1, and we will prove this for the important case when $n = 3$ in one of the following lectures. I recall that orthogonal matrices with determinant 1 represent rotations. This explains the importance of having a formula for all of them.