Archimedes and Pi

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Introduction

Proposition 3 of Archimedes' *Measurement of a Circle* states that π is less than 22/7 and greater than 223/71. The approximation $\pi_a \approx 22/7$ is referred to as Archimedes Approximation and is very good. It has been reported that a 2000 B.C. Babylonian approximation is $\pi_b \approx 25/8$. We will compare these two approximations. The author, in the spirit of idiot's advocate, will venture his own approximation of $\pi_c \approx 19/6$. The Babylonian approximation is good to one part in 189, the author's, one part in 125, and Archimedes an astonishing one part in 2484.

Archimedes' approach is to circumscribe and inscribe regular n-gons around a unit circle. He begins with a hexagon and repeatedly subdivides the side to get 12, 24, 48 and 96-gons. The semi-circumference of these polygons converge on π from above and below.

In modern terms, Archimede's derives and uses the cotangent half-angle formula,

$$\cot x/2 = \cot x + \csc x.$$

In application, the cosecant will be calculated from the cotangent according to the (modern) identity,

$$\csc^2 x = 1 + \cot^2 x$$

Greek mathematics dealt with ratio's more than with numbers. Among the often used ratios are the proportions among the sides of a triangle. Although Greek mathematics is said to not know trigonometric functions, we shall see how conversant it was with these ratios and the formal manipulation of ratios, resulting in a theory essentially equivalent to that of trigonometry.

For the circumscribed polygon

We use the notation of the Dijksterhuis translation of Archimedes. Circumscribe a hexagon around a circle centered at E. Let Γ be a point of tangency between the circle and the hexagon. Let Z be a vertex of the face bisected by Γ. Consider the right triangle $\triangle Z\Gamma E$. Let H be the point on

ΓZ such that HE bisects the angle $\angle ZE\Gamma$. The segment $H\Gamma$ is the half side of a circumscribed 12-gon. Continuing, let Θ be the point on ΓH such that ΘE bisects $\angle HE\Gamma$, K be the point on $\Theta\Gamma$ such that KE bisects $\angle \Theta E\Gamma$, and Λ be the point of $K\Gamma$ such that ΛE bisects $KE\Gamma$. The segments $\Theta\Gamma$, $K\Gamma$ and $\Lambda\Gamma$ are half sides of circumscribed 24, 48 and 96-gons, respectively.

Archimedes states the following ratio's for the hexagon,

$$\begin{array}{rcl} (E\Gamma, \Gamma Z) & > & (265, 153) \\ (ZE, \Gamma Z) & = & (306, 153) \end{array}$$

These two ratios approximate $\cot(\pi/6)$ and $\csc(\pi/6)$, respectively. We do not how he arrived at this excellent approximation of $\cot(\pi/6) = \sqrt{3}$ by a rational.

Archimedes then derives the cotangent half angle formula starting from Proposition 3 of Euclid's *Elements*, Book VI.

Lemma 1 (Euclid VI(3)) *Let $\triangle Z\Gamma E$ be a right triangle, with the right angle at Γ. Extend from E a bisector meeting $Z\Gamma$ at H. Then,*

$$(ZH, \Gamma H) = (ZE, \Gamma E).$$

where the notation (A, B) is the ratio A to B.

Proof: A proof does not appear in the original.

Let $E\Gamma$ be the radius of a circle with center E, and $Z\Gamma$ the tangent at Γ. Let X be the intersection of EZ with the circle. Construct a tangent to the circle at X and let H be the intersection of this tangent with $Z\Gamma$. Triangle $\triangle Z\Gamma E$ is congruent to $\triangle ZXH$. Therefore $(ZE, \Gamma E) = (ZH, HX)$ and $HX = \Gamma H$. □

Lemma 2 (Cotangent Half Angle Formula) *In the notation of the previous lemma,*

$$(\Gamma E, \Gamma H) = (ZE + E\Gamma, \Gamma Z).$$

Proof: To Euclid's equation, Archimedes applies the operation of *componendo* (Euclid V, Proposition 18),

$$(ZH, \Gamma H) = (ZE, \Gamma E) \Rightarrow (ZH + \Gamma H, \Gamma H) = (ZE + \Gamma E, \Gamma E)$$

and $ZH + \Gamma H = Z\Gamma$; then the operation of *permutando* (Euclid V, Proposition 16),

$$(Z\Gamma, \Gamma H) = (ZE + \Gamma E, \Gamma E) \Rightarrow (\Gamma E, \Gamma H) = (ZE + \Gamma E, Z\Gamma).$$

□

N.B.: Let the angle $\angle ZE\Gamma$ be θ. We transcribe the above lemma to modern notation, thus seeing how it is a half angle formula.

$$(\Gamma E, \Gamma H) = \Gamma E / \Gamma H = \cot \theta/2$$

and

$$(ZE + \Gamma E, Z\Gamma) = \frac{ZE + \Gamma E}{Z\Gamma} = \csc\theta + \cot\theta$$

Lemma 3 (Pythagorean cosecant formula) *In the notation of the above two lemmas,*

$$((HE)^2, (\Gamma H)^2) = ((\Gamma E)^2 + (\Gamma H)^2, (\Gamma H)^2)$$

Proof: HE is the hypotenuse of the right triangle $\triangle HE\Gamma$. Apply the Pythagorean formula, Euclid book 1, Proposition 47. □

N.B.: Replacing the ratios of the previous lemma with modern notation,

$$\csc^2\theta = 1 + \cot^2\theta.$$

The above lemma does not appear explicitly in the proof. Rather it is implicit in the next three lines of calculation,

$$\begin{array}{rcl} (\Gamma E, \Gamma H) & > & (571, 153) \\ ((EH)^2, (\Gamma H)^2) & > & (349450, 23409) \\ (EH, \Gamma H) & > & (591\ 1/8, 153) \end{array}$$

These are, respectively, approximations for $\cot(\pi/12), \csc^2(\pi/12)$ and $\csc(\pi/12)$. In the orignal, Archimedes sometimes used equality when inequality is correct. Although his calculation of the square root were properly rounded, the notation did not always highlight this.

The proof continues by iteratively bisecting three more times the angle defining the polygon, defining points on Θ, K and Λ, the vertices adjacent to the tangency at Γ on a 24, 48 and 96-gon, respectively. In the original, the following ratio are given. We have added an annotation column giving the trignometric interpretation.

$$\begin{array}{rcl} (E\Gamma, \Gamma\Theta) & > & (1162\ 1/2, 153) \\ (E\Theta, \Theta\Gamma) & > & (1172\ 1/8, 153) \\ (E\Gamma, \Gamma K) & > & (2334\ 1/4, 153) \\ (EK, K\Gamma) & > & (2339\ 1/4, 153) \\ (E\Lambda, \Lambda\Gamma) & > & (4673\ 1/2, 153) \end{array}$$

The proof concludes noting that half the circumference of the circle is less than half the perimeter of the circumscribed 96-gon. The ratio is then rounded,

$$\pi < 96 \cdot (153, 4673\ 1/2) < 22/7$$

For the inscribed polygon

Let ΓB be a face of the inscribed polygon. Construct a diameter of the circle it one end at Γ and the other call A. Then $\triangle \Gamma BA$ is a right triangle. Call H the point at which the bisector of angle

$\angle BA\Gamma$ crosses the circle on the arc between Γ and B. Segment ΓH is the face of an inscribed polygon of twice the number of faces as the original polygon. Continuing, bisect $\angle HA\Gamma$ to locate point Θ, bisect $\angle \Theta A\Gamma$ to locate point K, and finally $\angle KA\Gamma$ to locate point Λ. If ΓB is the face of a hexagon, then $H\Gamma$, $\Theta\Gamma$, $K\Gamma$, and $\Lambda\Gamma$ are faces of 12, 24, 48 and 96-gons.

The approximation to $\cot(\angle BA\Gamma)$ is given,

$$(BA, B\Gamma) < (1351, 780).$$

Archimedes then derives the formula,

$$(AH, H\Gamma) = (A\Gamma + AB, B\Gamma).$$

Expanding this out in modern notation, we again see the half angle formula for cotangent. Archimedes next applies the formula giving an approximation to $\cot(\angle HA\Gamma)$,

$$(HA, H\Gamma) < (2911, 780).$$

Using the cosecant formula, without statement, Archimedes approximates,

$$(A\Gamma, \Gamma H) < (3013\ 3/4, 780).$$

Assuming now that his reader is intelligently following his iteration, successive cotangents and cosecants are given for the remaining bisections,

$$\begin{array}{rcl} (A\Theta, \Theta\Gamma) & < & (1823, 240) \\ (A\Gamma, \Gamma\Theta) & < & (1838\ 9/11, 240) \\ (AK, K\Gamma) & < & (1007, 66) \\ (A\Gamma, \Gamma K) & < & (1009\ 1/6, 66) \\ (A\Lambda, \Lambda\Gamma) & < & (2016\ 1/6, 66) \\ (A\Gamma, \Gamma\Lambda) & < & (2017\ 1/4, 66) \end{array}$$

, Since $A\Gamma$ is the diameter of a unit radius circle, the semi-perimeter of the inscribed 96-gon is,

$$(96 \cdot 66, 2017\ 1/40) = (6336, 2017\ 1/4).$$

Hence,

$$\pi > \frac{6336}{2017\ 1/4} > 3\ 10/71.$$

Thus completing the demonstration.

Error and convergence

Archimedes did not talk of convergence properties. He did consider error, since the extraction of square roots is approximate and he carefully rounded appropriately, the lower bound being subject to rounding down, the upper bound to rounding up.

The semiperiemeters of circumscribed and inscribed n-gons, P_n and p_n respectively, are given by,

$$\begin{aligned} P_n &= n/\cot(\pi/n) \\ p_n &= n/\csc(\pi/n) \end{aligned}$$

Using the cosecant identity,

$$1/p_n^2 = 1/n^2 + 1/P_n^2.$$

or

$$P_n - p_n = \frac{(P_n p_n)^2}{n^2(P_n + p_n)}$$

When $n \geq 6$, we have the bounds $3 < P_n < 2\sqrt{3}$ and $3 < p_n < \pi$. This gives a simpler inequality for the error,

$$P_n - p_n < 2(\pi/n)^2.$$

Appendix: Sources

Archimedes studied in Alexandria (269–263 B.C.) and for the rest of his life communicated his results by letter to his friends there. He generally first sent results with encouragement to find proofs, and later full treatments including proofs. None of this has survived. All modern and medieval copies of his work seem to come through a 10-th century compendium of his work.

In the 12-th century, this compendium was taken apart, erased and the parchment reused. Such a practice was common and the resulting book is called a Palimpsest. Being a religious text it became sacred and for the next 600 years was read from and cared for by the Greek Orthodox adherents at the monastery of Mar Saba in the Dead Sea. It was moved to the library of the Greek Patriarch in the Christian quarter of Jerusalem and then in the early 1800's to the Metochion in Constantinople, a daughter house of the Church of the Holy Sepulchre.

In 1907 Johan Heiburg discovered the manuscript by reading and identifying the erased text. The manuscript disappeared in the mid 20-th century, to turn up mysteriously at auction. It was sold at Christie's of New York on October 29, 1998 to an anonymous collector for two million dollars. The collector loaned the Archimedes Palimpsest to the Walters Art Gallery in Baltimore, Maryland where it now resides. This history is based on Heath and the Walters Art Gallery web site.

References

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