Decidability

Decidable Problems About Regular Languages

The Acceptance Problem for DFA

Define A_{DFA} to be:

$$\{\langle B, w\rangle \mid B \text{ is a DFA that accepts input string } w\}.$$

Theorem. **A_{DFA} is decidable.**

Proof A Turing machine can, given an input x, try to decode x into an NFA B and a string w. If the decoding is successful then it can test whether B accepts w by **simulating B on w.** ∎

How This Can Be Done

- After checking the legitimacy of encoding, our Turing machine writes on its second tape the input w (as an encoded form).
- Our machine starts simulating M, using the second tape as the tape of M by looking up information about M's action in the first tape and using a tape symbol encoding scheme consistent with the input x.

  When M terminates, our machine terminates accordingly.

The Acceptance Problem for NFA

Define A_{NFA} to be:

$$\{\langle B, w\rangle \mid B \text{ is an NFA that accepts input string } w\}.$$

Theorem. **A_{NFA} is decidable.**

Proof Given an input x, try to decode x into an NFA B and a string w. If "successful" then:

1. Convert B to a DFA C.
2. Run the machine for A_{DFA} on $\langle C, w\rangle$. If the machine accepts, then **accept**; otherwise **reject**.

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The Acceptance Problem for Regular Exp.

Define A_{REX} to be:

$$\{\langle R, w\rangle \mid R \text{ is a regular expression that produces } w\}.$$

Theorem. **A_{REX} is decidable.**

Proof Given an input x, try to decode x into a regular expression R and a string w. If "successful" then:

1. Convert R to a DFA C.
2. Run the machine for A_{DFA} on $\langle C, w\rangle$. If the machine accepts, then **accept**; otherwise **reject**.

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The Emptiness Problem for DFA

Define $E_{\mathrm{DFA}} = \{\langle A \rangle \mid A$ is a DFA that accepts no string $\}$.

Theorem. **E_{DFA} is decidable.**

Proof Given an input x, try to decode a DFA A out of x. If "successful" then:

1. **Mark the start state** of A.
2. **Repeat until no new states are marked:**
   - Mark any unmarked state that has **a transition from a marked state**
3. Accept if **no final state is marked**; reject otherwise.

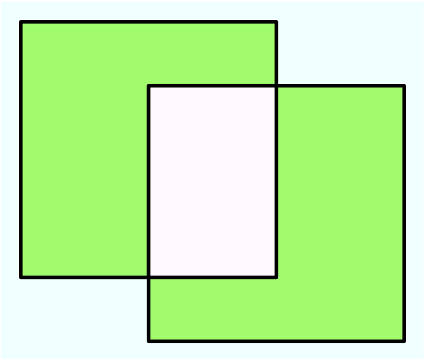
The Equivalence Problem for DFA

Define EQ_{DFA} to be:
$\{\langle A, B\rangle \mid A \text{ and } B \text{ are DFA and accept the same language } \}$.
Theorem. ***EQ_{DFA} is decidable.***

Proof Given a string x, try to decode x into a pair of DFAs A and B. If "successful" then construct a DFA C that accepts the **symmetric difference** of $L(A)$ and $L(B)$,

$$(L(A) \cap \overline{L(B)}) \cup (\overline{L(A)} \cap L(B)),$$

and test the emptiness of $L(C)$. ∎



The Acceptance Problem for CFG

Define $A_{\mathrm{CFG}} = \{\langle G, w\rangle \mid G$ is a CFG that generates $w\}$.

Theorem. A_{CFG} **is decidable.**

Proof Given an input x, try to decode x into a CFG G and a string w. If "successful" then:

1. Convert G to an equivalent Chomsky normal form grammar G'.
2. List all derivations with $2n - 1$ steps, where $n = |w|$.
3. If any of the listed derivations generate w, then **accept**; otherwise, reject.

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The Emptiness Problem for CFG

Define $E_{\mathrm{CFG}} = \{\langle G \rangle \mid G$ is a CFG such that $L(G) = \emptyset\}$.

Theorem. **E_{CFG} is decidable.**

Proof Given x, first try to decode a grammar G out of it. If "pass" then test the ability of generating terminal strings:

1. **Mark all the terminals**.
2. Repeat the following until no new symbols are marked:
   – Mark any variables A with **a production $A \to w$ such that all symbols in w are marked**.
3. **Accept** if the start symbol is marked; **reject** otherwise.

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Context-Free Languages are Decidable

Theorem. **Every context-free language is decidable.**

Simulation of a PDA may not halt.

Proof Use the machine M for A_{CFG}. Let G be a fixed CFG. The machine for $L(G)$, on input w,

1. run $\langle G, w \rangle$ on M, and
2. **accepts** if M accepts and **rejects** otherwise.

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